## SECTION 4.6: LINEARIZATION

**RECALL:** If $f$ is differentiable at $x = a$. then $f$ is locally linear at $x = a$ meaning that as $x$ gets 'close to' $a$, the curve $y = f(x)$ and the tangent line at $x = a$ become indistinguishable. In symbols,

$$f(x) \approx f'(a)(x - a) + f(a)$$

**DEFINITION:** If $f$ is differentiable at $x = a$, the **linearization of** $f$ **at** $x = a$, denoted $L_a(x)$ is:

$$L_a(x) = f'(a)(x - a) + f(a)$$

**NOTE:** $L_a(x)$ is precisely the tangent line to $y = f(x)$ at $x = a$.

**EXAMPLE 1:** Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 0$ and use it to approximate $\sqrt{1.01}$.

$L_0(x) = f'(0)(x - 0) + f(0)$. We have $f(0) = \sqrt{1+0} = \sqrt{1} = 1$ so we work to find $f'(0)$.

$$f'(x) = D_x\left[\sqrt{1+x}\right] = D_x\left[(1+x)^{1/2}\right] = \frac{1}{2}(1+x)^{-1/2}D_x[1+x] = \frac{1}{2}(1+x)^{-1/2}(1) = \frac{1}{2\sqrt{1+x}}.$$

Hence, $f'(0) = \dfrac{1}{2\sqrt{1+0}} = \dfrac{1}{2}$. Therefore, we have $L_0(x) = \dfrac{1}{2}(x-0) + 1$ or $L_0(x) = \dfrac{1}{2}x + 1$.

To use $L_0(x)$ to approximate $\sqrt{1.01}$, we note $\sqrt{1.01} = \sqrt{1+0.01} = f(0.01)$.

Hence, $\sqrt{1.01} = \sqrt{1+0.01} = f(0.01) \approx L_0(0.01) = \dfrac{1}{2}(0.01) + 1 = 1.005$.

**NOTE:** The actual retail value of $\sqrt{1.01} = 1.0049875621\ldots$

**EXAMPLE 2:** Use $L_0(x)$ from the previous example to approximate $\sqrt{0.8}$.

Compare your answer with the actual retail value of $\sqrt{0.8}$.

$\sqrt{0.8} = \sqrt{1+(-0.2)} = f(-0.2) \approx L_0(-0.2) = \dfrac{1}{2}(-0.2) + 1 = 0.9$. Hence, $\sqrt{0.8} \approx 0.9$.

The actual retail value of $\sqrt{0.8} = 0.894427\ldots$

**EXAMPLE 3:** Find the linearization of $f(x) = \cos(x)$ at $x = \dfrac{\pi}{2}$ and use it to approximate $\cos(88^\circ)$.

$L_{\frac{\pi}{2}}(x) = f'\left(\dfrac{\pi}{2}\right)\left(x - \dfrac{\pi}{2}\right) + f\left(\dfrac{\pi}{2}\right)$. We have $f\left(\dfrac{\pi}{2}\right) = \cos\left(\dfrac{\pi}{2}\right) = 0$, so we work to find $f'\left(\dfrac{\pi}{2}\right)$.

We have $f'(x) = D_x[\cos(x)] = -\sin(x)$ so $f'\left(\dfrac{\pi}{2}\right) = -\sin\left(\dfrac{\pi}{2}\right) = -1$. Hence,

$$L_{\frac{\pi}{2}}(x) = f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = (-1)\left(x - \frac{\pi}{2}\right) + 0 = -x + \frac{\pi}{2}$$

To use $L_{\frac{\pi}{2}}(x)$ to approximate $\cos(88^\circ)$, we need to convert $88^\circ$ to radians: $88^\circ = \dfrac{88^\circ\pi}{180^\circ} = \dfrac{22\pi}{45}$. Hence,

$$\cos(88^\circ) = \cos\left(\frac{22\pi}{45}\right) \approx L_{\frac{\pi}{2}}\left(\frac{22\pi}{45}\right) = -\frac{22\pi}{45} + \frac{\pi}{2} = \frac{\pi}{90}$$

Hence, $\cos(88^\circ) \approx \dfrac{\pi}{90} \approx 0.0349065850$.

**NOTE:** The actual retail value of $\cos(88^\circ) = 0.0348994967025009\ldots$

**EXAMPLE 4: (VIDEO)** Find the linearization of $f(x) = \sin(x)$ at $x = 0$ and use it to approximate $\sin(3°)$ Compare your answer with the actual retail value of $\sin(3°)$.

Ans: $L_0(x) = f'(0)(x-0) + f(0) = \ldots = x$. Next, $3° = 3° \dfrac{\pi}{180°} = \dfrac{\pi}{60}$ radians:

Hence, $\sin(3°) = \sin\left(\dfrac{\pi}{60}\right) \approx L_0\left(\dfrac{\pi}{60}\right) = \dfrac{\pi}{60}$.

Note that $\dfrac{\pi}{60} = 0.052359\ldots$ while $\sin(3°) = 0.05233\ldots$

**EXAMPLE 5: (VIDEO)** For $f(x) = (1+x)^p$, derive a formula for the linearization at $x = 0$.

Ans: $L_0(x) = f'(0)(x-0) + f(0) = \ldots = px + 1$.

**NOTE:** In the case $p = \dfrac{1}{2}$, we get $\sqrt{1+x} = (1+x)^{1/2} \approx \dfrac{1}{2}x + 1$ which we saw in earlier examples.

**EXAMPLE 6:** The the amount of money (dollars) in an account after 10 years if invested at an annual interest rate $r$ compounded monthly is given by:

$$A(r) = 300\left(1 + \frac{r}{12}\right)^{120}$$

1. Find the linearization of $A$ about $r = 0$.

   $L_0(r) = A'(0)(r-0) + A(0)$. We have $A(0) = 300\left(1 + \dfrac{0}{12}\right)^{120} = 300$ so work to find $A'(0)$.

   $$A'(r) = 300(120)\left(1 + \frac{r}{12}\right)^{119} D_r\left[1 + \frac{r}{12}\right] = 300(120)\left(1 + \frac{r}{12}\right)^{119} \frac{1}{12} = 3000\left(1 + \frac{r}{12}\right)^{119}$$

   Hence, $A'(0) = 3000$. So our Linearization is: $L_0(r) = A'(0)(r-0) + A(0) = 3000(r-0) + 300 = 3000r + 300$.

2. Find and interpret $L(0.02)$ and $L(0.05)$.

   $L(0.02) = 3000(0.02) + 300 = 360$.

   If the annual interest rate were 2%, we would expect approximately \$360 in the account after 10 years.

   $L(0.05) = 3000(0.05) + 300 = 450$.

   If the annual interest rate were 5%, we would expect approximately \$450 in the account after 10 years.

   **NOTE:** Since $A'(0) = 3000$, $A$ is increasing at a rate of \$3000 per $1 = 100\%$ increase in $r$.

   Said differently, $A$ increases \$30 for each increase in percentage point.

   This is exactly what we're seeing in these linearization results!

# DIFFERENTIALS

**RECALL:** If $f$ is differentiable at $x = a$, then $f(a+h) \approx f(a) + f'(a)\,h$.

In this section, we give the quantities $h$ and $f'(a)\,h$ names.

**DEFINITIONS:** Suppose $y = f(x)$.

- **RECALL:** $x$ is called the **independent** variable and $y$ is called the **dependent** variable.

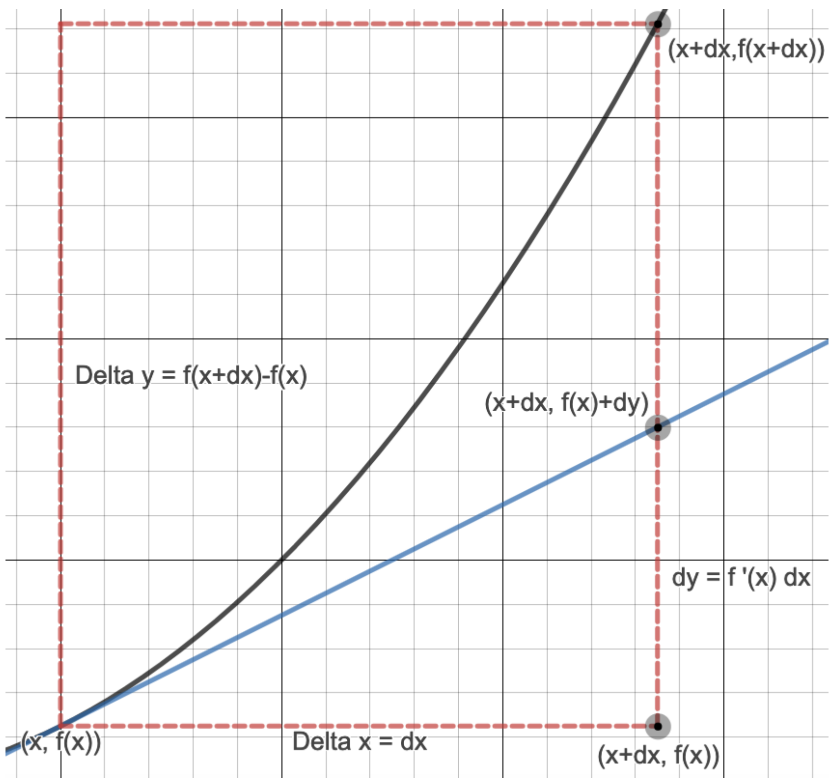
- The quantity '$dx$' is called the **increment of the independent variable** and can have the value of any nonzero real number. That is, $dx$ represents the change in $x$, so, in other words: $dx = \Delta x$.

  **NOTE:** In the formulation $f(a+h) \approx f(a) + f'(a)\,h$, $h = \Delta x = dx$.

- The quantity '$dy$' is called the **differential of the dependent variable** and is **defined** as: $dy = f'(x)\,dx$.

  **NOTE:** In the formulation $f(a+h) \approx f(a) + f'(a)\,h$, $f'(a)\,h = dy$.

Schematically:




Hence:

- $\Delta x = dx$ = change in inputs on $x$-axis.
- $\Delta y = f(x+dx) - f(x)$ change in outputs on the graph of $f$.
- $dy = f'(x)\,dx$ change in outputs on tangent line.

**NOTE:** The Leibniz notation for derivative: $f'(x) = \dfrac{dy}{dx}$ can also be used to motivate the notation for differential:

$$f'(x) = \frac{dy}{dx} \iff dy = f'(x)\,dx$$

**APPROXIMATION THEOREM (DIFFERENTIAL VERSION):**

$$f(x+dx) \approx f(x) + dy = f(x) + f'(x)\,dx \text{ or, equivalently, } \Delta y = f(x+dx) - f(x) \approx dy = f'(x)\,dx$$

**EXAMPLE 7:** Suppose $f'(5) = -2$.

1. If $dx = 0.2$, find $dy$ when $x = 5$.

   $dy = f'(x)\,dx$ so $dy = f'(5)\,dx = (-2)(0.2) = -0.4$.

2. Use the fact that $f(5) = 6$ to approximate $f(5.2)$ using differentials.

   Since $5.2 - 5 = 0.2$, we have $dx = 0.2$ and from the previous problem, we know $dy = -0.4$.

   Since $f(x+dx) \approx f(x) + dy$, $f(5.2) = f(5+0.2) \approx f(5) + dy = 6 + (-0.4) = 5.6$.

3. Use differentials to approximate $f(4.5)$ using what is known about $f(5)$ and $f'(5)$.

   Since $4.5 - 5 = -0.5$, we have $dx = -0.5$. Hence, $dy = f'(5)\,dx = (-2)(-0.5) = 1$.

   We get: $f(4.5) = f(5 + (-0.5)) \approx f(5) + dy = 6 + 1 = 7$.

**EXAMPLE 8:** Let $f(x) = \sqrt{x}$.

1. Find an expression for $dy$.

   Since $dy = f'(x)\,dx$ and $f'(x) = D_x[\sqrt{x}] = \dfrac{1}{2\sqrt{x}}$, we have $dy = \dfrac{1}{2\sqrt{x}}\,dx$

2. Use differentials to approximate $\sqrt{11}$ using the fact $\sqrt{9} = 3$.

   Since we are basing our approximation at $x = 9$, we have $dx = 11 - 9 = 2$.

   When $x = 9$ and $dx = 2$, get $dy = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2\sqrt{9}}(2) = \dfrac{1}{3}$.

   Hence, $\sqrt{11} = \sqrt{9+2} \approx \sqrt{9} + dy = 3 + \dfrac{1}{3} = 3.\overline{3}$.

   **NOTE:** The actual retail value of $\sqrt{11} = 3.31662\ldots$.

**EXAMPLE 9: (VIDEO)** Let $f(x) = \sqrt[3]{x}$.

1. Find an expression for $dy$.

   Ans: $dy = \dfrac{1}{3}x^{-2/3}\,dx$.

2. Use differentials to approximate $\sqrt[3]{6}$ using the fact $\sqrt[3]{8} = 2$.

   Ans: $dy = -\dfrac{1}{6}$ so $\sqrt[3]{6} = \sqrt[3]{8 + (-2)} \approx \sqrt[3]{8} + dy = \ldots = 1.8\overline{3}$.

Since differentials are based on derivatives, they enjoy the same properties.

**PROPERTIES OF DIFFERENTIALS:** Let $u$ and $v$ be differentiable functions of $x$ and $k$ a constant.

- **CONSTANT MULTIPLE RULE:** $d[k\,u] = k\,du$.
- **SUM AND DIFFERENCE RULE:** $d[u \pm v] = du \pm dv$.
- **PRODUCT RULE:** $d[uv] = u\,dv + v\,du$
- **QUOTIENT RULE:** $d\left[\dfrac{u}{v}\right] = \dfrac{v\,du - u\,dv}{v^2}$

**EXAMPLE 10:** The density of a substance, $\rho$, is calculated by dividing its mass, $m$, by its volume, $V$: $\rho = \dfrac{m}{V}$.

A scientist collects $5 \pm 0.5$ mL of a substance and determines its mass to be $68.2 \pm 0.1$g.

She computes the density as: $\rho = \dfrac{68.2\,\text{g}}{5\,\text{mL}} = 13.64$ g / mL.

1. Use the equation $\rho = \dfrac{m}{V}$ to relate the differentials $d\rho$, $dm$, and $dV$.

   Using the Quotient Rule, we get: $d\rho = \dfrac{V\,dm - m\,dV}{V^2}$.

2. $d\rho$ helps us estimate the **propagated error** in using our measurements of mass and volume, which have uncertainties, to compute density. Using $m = 68.2$ g, $dm = \pm 0.1$ g, $V = 5$ mL and $dV = \pm 0.5$ mL we get:

$$d\rho = \frac{V\,dm - m\,dV}{V^2} = \frac{(5\,\text{mL})(\pm 0.1\,\text{g}) - (68.2\,\text{g})(\pm 0.5\text{ mL})}{(5\,\text{mL})^2} = \frac{(\pm 0.5\,\text{g mL}) - (\pm 34.1\,\text{g mL })}{25\,\text{mL}^2}$$

   Since we don't know which of + or − is involved with ± and we're trying to determine error, we typically choose the largest possible combination. Hence we simplify $(\pm 0.5\,\text{g mL}) - (\pm 34.1\,\text{g mL })$ by assuming the errors stack and write this as $\pm(0.5 + 34.1)\,\text{g mL} = \pm 34.6\,\text{g mL}$. Hence,

$$d\rho = \frac{(\pm 0.5\,\text{g mL}) - (\pm 34.1\,\text{g mL })}{25\,\text{mL}^2} = \pm\frac{34.6\,\text{g mL}}{25\,\text{mL}^2} = \pm 1.384\,\frac{\text{g}}{\text{mL}}$$

   This means the propagated error in our density, $\rho$, is $\pm 1.384$ g/mL. In other words, $\rho = 13.64 \pm 1.384$ g/mL.

3. Find and interpret $\dfrac{d\rho}{\rho}$.

   We find $\dfrac{d\rho}{\rho} = \pm\dfrac{1.384\,\text{g/mL}}{13.64\,\text{g/mL}} \approx 0.1015 = 10.15\,\%$. This means our **percent relative error** is roughly $10.15\,\%$.

**HOMEWORK:** Section 4.6: 21 - 69 every other odd, 52*